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No. 6

Theoretical Wave Studies

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(Columbia University)

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Technical Report No. 6

by

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Part 1 - On the Existence of Uller's Waves

ABSTRACT

A reexamination of a general theory of wave propagation proposed by Uller is carried out. The conclusion is reached that his hypothesis has not been placed on a firm physical foundation and cannot be considered valid in interpreting seismic data.

* * * * * *

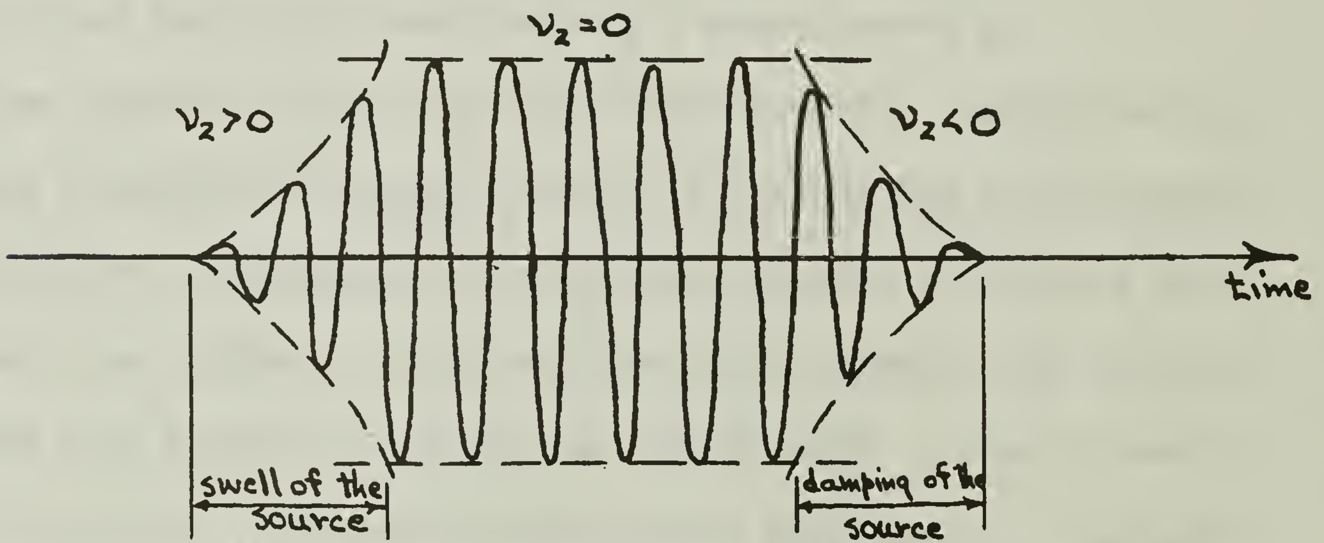
In recent years a general wave theory has been proposed by Karl Uller. Uller thought that the existing theory was inadequate in its explanation of several physical phenomena, particularly with respect to elastic surface wave propagation. He proposed a general wave theory which would be able to explain all observable types of waves. This theory was based on a single wave function, equation (1), employed in all subsequent work. In most of his studies a non-dispersive, non-viscous, isotropic, solid medium has been considered.

His work has received attention from a number of seismologists, and it has been thought by some that his method can lead to an interpretation of many of the presently unexplained features of seismograms. It is the purpose of this paper to show that the general wave function, ψ , (equation 1), is not the solution of a wave equation and that consequently his theory cannot be considered valid in interpreting data obtained from elastic wave propagation.

Uller's theory is based on the premise that the funadmental wave function is given by

$$\psi = e^{-(\nu_2 t - \Phi_2)}\left\{A_1 \cos(\nu_1 t - \Phi_1) + A_2 \sin(\nu_1 t - \Phi_1)\right\} \qquad (1)$$

where ν_1 is the frequency of the source, ν_2 is the swell or damping of the source, and A_1, A_2, Φ_1, and Φ_2 are functions of the space coordinates. The symbol, ν_2, does not represent absorption in the elastic medium but relates directly to the source. The meaning of ν_2 is most easily seen from the following diagram[1], which represents a wave sent out by the source.



The source, operating at a constant frequency, builds up to a particular amplitude (swell of the source, $\nu_2 > 0$) and then decreases back to its undisturbed state (damping of the source, $\nu_2 < 0$). It is difficult to grasp the physical significance of such a source and the function, ψ, related to that source.

As Uller spends a great deal of time on plane wave motion, it is considered worthwhile to show explicitly that even in this case his proposal is incorrect. Following his line of reasoning[2], the plane waves are obtained when A_1 and A_2 are constants and when Φ_1 and Φ_2

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1. Uller, K., Gerlands Beiträge zur Geophysik, 20, 131, (1928)
2. Uller, K., Gerlands Beiträge zur Geophysik, 18, 402, (1927)

are given by $\mathbf{w}_1 \cdot \mathbf{r}$ and $\mathbf{w}_2 \cdot \mathbf{r}$ respectively, where $\mathbf{r}$ is the topographic radius and where $\mathbf{w}_1$ and $\mathbf{w}_2$ are determined from the constants of the medium. Then according to Uller, there will be two travelling quantities $\nu_1 t - \mathbf{w}_1 \cdot \mathbf{r}$ and $\nu_2 t - \mathbf{w}_2 \cdot \mathbf{r}$. These two travelling quantities will be propagated with velocities $\frac{\nu_1}{w_1}$ and $\frac{\nu_2}{w_2}$ respectively; and they will have an angle, θ, between their directions of propagation where θ is the angle between $\mathbf{w}_1$ and $\mathbf{w}_2$.

It will now be shown that the interpretation given to ψ in the above paragraph is incorrect. As stated above, for plane waves Uller's ψ function is given by

$$\psi = e^{-(\nu_2 t - \mathbf{w}_2 \cdot \mathbf{r})} \left\{ A_1 \cos(\nu_1 t - \mathbf{w}_1 \cdot \mathbf{r}) + A_2 \sin(\nu_1 t - \mathbf{w}_1 \cdot \mathbf{r}) \right\}$$

or

$$\psi = e^{-(\nu_2 t - \mathbf{w}_2 \cdot \mathbf{r})} \left\{ B \cos(\nu_1 t - \mathbf{w}_1 \cdot \mathbf{r} - \delta) \right\} \qquad (2)$$

where A_1, A_2, B and δ are constants. This function actually represents a single propagation with velocity intermediate between $\frac{\nu_1}{w_1} + \frac{\nu_2}{w_2}$ and $\frac{\nu_1}{w_1} - \frac{\nu_2}{w_2}$, and it is propagated in the plane determined by $\mathbf{w}_1$ and $\mathbf{w}_2$.

This can be followed through very simply both algebraically and geometrically for the case where $\mathbf{w}_1 \cdot \mathbf{r} = b_1 x$ and $\mathbf{w}_2 \cdot \mathbf{r} = b_2 y$. There we may write

$$\psi = B e^{-\nu_2 \left(t - \frac{y}{c_2}\right)} \cos\left\{\nu_1\left(t - \frac{x}{c_1} - \epsilon\right)\right\} \qquad (3)$$

where $c_2 \equiv \frac{\nu_2}{b_2}$, $c_1 \equiv \frac{\nu_1}{b_1}$, and $\epsilon \equiv \frac{\delta}{\nu_1}$. Let us now investigate the map of this function at a time, t'.

ψ will have zeros given by

$$\nu_1\left(t' - \frac{x}{c_1} - \epsilon\right) = (2n+1)\frac{\pi}{2} \quad \text{where } n = 1, 2, 3, \cdots$$

ψ will decrease exponentially along the positive Y-direction and will oscillate along the X-direction. The positions where ψ has the value B are indicated by ● in Figure 1:

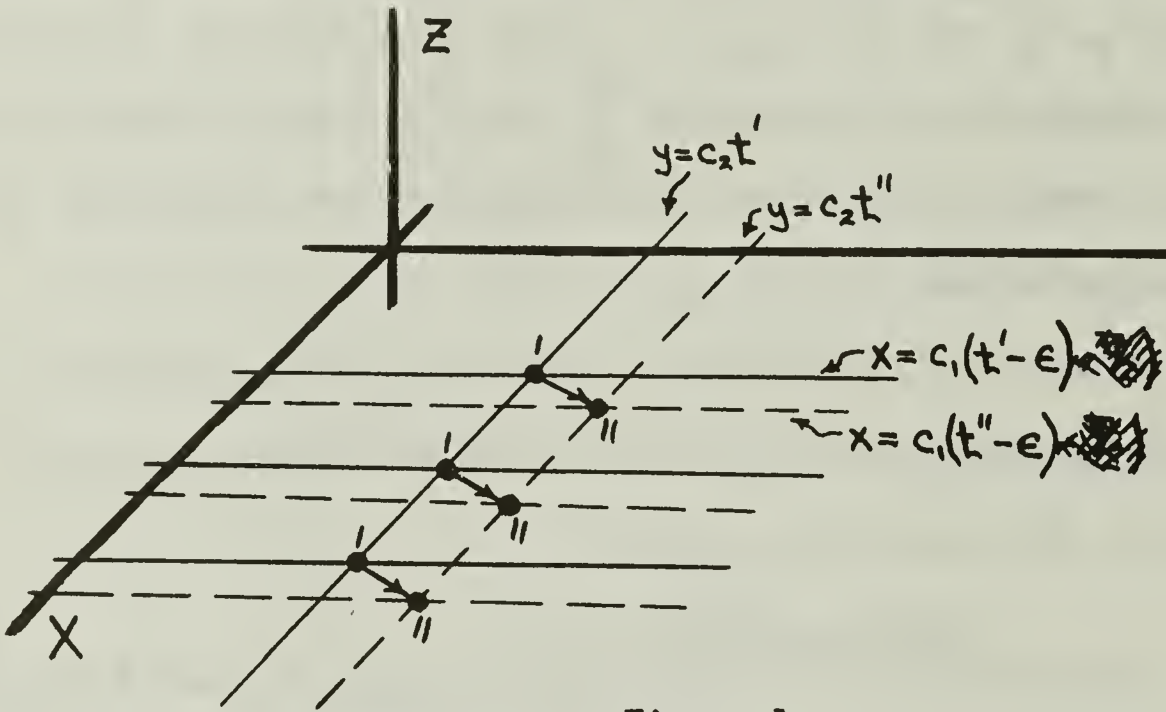

Figure 1

At a later instant t" the positions of the zeros will be given by

$$\nu_1\left(t'' - \frac{x}{c_1} - \epsilon\right) = (2n+1)\pi/2$$

and the marked positions have moved as indicated. It is easy to see now that the direction of propagation will be given by $\tan^{-1}(c_1/c_2)$ from the X-axis in the XY plane and that the velocity of propagation is $\sqrt{c_1^2+c_2^2}$. The above relations can be easily generalized for arbitrary directions of $\underset{\sim}{w}_1$ and $\underset{\sim}{w}_2$, giving a velocity of propagation of $\sqrt{c_1^2+c_2^2+2c_1c_2\cos\theta}$. The direction of propagation will be at an angle $\tan^{-1}(c_2\sin\theta/c_1+ c_2\cos\theta))$ measured from $\underset{\sim}{w}_1$ in the plane determined by $\underset{\sim}{w}_1$ and $\underset{\sim}{w}_2$.

Again using equation (3) let us make the transformation,

$$x = x'\cos\varphi - y'\sin\varphi \qquad \text{where } \varphi = \tan^{-1}\frac{c_1}{c_2}$$
$$y = x'\sin\varphi + y'\cos\varphi$$

or

$$x = \frac{c_1 x'}{\sqrt{c_1^2 + c_2^2}} + \frac{c_2 y'}{\sqrt{c_1^2 + c_2^2}}$$

$$y = \frac{c_2 x'}{\sqrt{c_1^2 + c_2^2}} - \frac{c_1 y'}{\sqrt{c_1^2 + c_2^2}}$$

Then we may write,

$$\psi = Be^{-\nu_2\left(t - \frac{x'}{\sqrt{c_1^2+c_2^2}} + \frac{c_1 y'}{c_2\sqrt{c_1^2+c_2^2}}\right)} \cos\nu_1\left(t - \frac{x'}{\sqrt{c_1^2+c_2^2}} - \frac{c_2 y'}{c_1\sqrt{c_1^2+c_2^2}} - \epsilon\right)$$

$$= Be^{-\frac{\nu_2 c_1}{c_2\sqrt{c_1^2+c_2^2}}} e^{-\nu_2\left(t - \frac{x'}{\sqrt{c_1^2+c_2^2}}\right)} \tag{4}$$

$$\times\left[\cos\frac{\nu_1 c_2 y'}{c_1\sqrt{c_1^2+c_2^2}}\cos\nu_1\left(t - \frac{x'}{\sqrt{c_1^2+c_2^2}} - \epsilon\right) + \sin\frac{\nu_1 c_2 y'}{c_1\sqrt{c_1^2+c_2^2}}\sin\nu_1\left(t - \frac{x'}{\sqrt{c_1^2+c_2^2}} - \epsilon\right)\right]$$

As before ψ is seen to represent a single propagation in the X'-direction with velocity $\sqrt{c_1^2+c_2^2}$; the terms in y' give the amplitude modulation of the wave.

Continuing to a more general discussion, it is noted that equation (1) does not satisfy the simple wave equation

$$\ddot{\varphi} = c^2 \nabla^2 \varphi$$

for the fundamental solution of this equation is of the form

$$\varphi = f(r - ct)$$

A wave of the form

$$\varphi = g(r_1 - c_1 t)\, h(r_2 - c_2 t)$$

cannot satisfy equation (5).

We may go on to say that Uller's wave will not satisfy <u>any</u> one dimensional wave equation. The reader may be referred to a recent article by Eckart[3] on this subject, where the general solution for one dimensional wave propagation is given by

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, e^{i(kx - H(k)t) - D(k)t}\, dk \tag{8a}$$

where

$$F(k) = \int_{-\infty}^{\infty} \psi_0(x)\, e^{-ikx}\, dx \tag{8b}$$

The integration is taken over real values of k and x. H(k) and D(k) are functions of k which depend upon the original differential equation. It is to be especially noticed that the space coordinate enters only in the imaginary part of the exponent. There is a single propagating quantity.

The only place where one meets a wave similar to that given by Uller is in surface wave propagation in an absorptive medium. There one obtains an expression of the form

$$\varphi = A e^{-\sigma t} e^{-ny} \cos(kx - ct) \tag{9}$$

However, in this case the exponential in time is due to the fact that the wave is being propagated in an absorptive medium; and the exponential in the y-coordinate is due to the fact that a free surface wave is being examined. They do not depend upon the original condition of the source. As his waves do depend upon an unusual source condition, the two waves cannot be compared.

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3. Eckart, C., Rev. Mod. Phys., <u>20</u>, 400, (1948)

The faults of Uller's method are indicated by his own application of the hypothesis to the differential equation of wave propagation in a gravitating medium. Upon substituting equation (1) into the differential equation, one finds that the arbitrary coefficients A_1 and A_2, are no longer arbitrary. Such an answer cannot be considered valid.

From the discussion in the preceding paragraphs, the interpretation of the wave function, ψ, becomes clear. It represents a single propagation, not a double propagation. Of greater importance is the fact that Uller does not refer the function, ψ, to any differential wave equation. The conditions imposed by such differential equations are in general not considered. Nevertheless, it is stated that all elementary types of wave phenomena can be represented by equation (1).

It is interesting to continue on now to an examination of some of Uller's comments on the present theory. In one of his first articles[4], the following statement is found: "Der Begriff der Gruppengeschwindigkeit aber, sofern es sich nicht um solche Interferenz handelt, ist eine mathematische Verlegenheitskonstruktion. Zu der wahren Erklärung konnte man nicht gelangen, weil man mit ungedämpften und einfachen Sinuswellen arbeitete." The concept of group velocity is not an ambarrassment to applied mathematics. It is a necessary and most useful tool in dispersive wave propagation.

In another section one finds the statement[5]: "Es wird erstens der Gültigkeitsbezirk des Snelliusschen Brechungsgesetzes als beschränkt erwiesen. . . . Ausserhalb des Snelliusschen Bezirkes gelten nicht die Sätze von Fermat, Malus, Hamilton, Helmholtz, und Kirchoff." These

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4. Uller, K., Gerlands Beiträge zur Geophysik, 18, 404, (1927)
5. ibid, 398

various laws have formed the basis of our present ideas on wave propagation, and any hypothesis which is contrary to them should be questioned severely.

Elsewhere the following statement is found[6]: "Es is schon lange bekannt, dass die Oberflächenwelle an einer Flüssigkeit, als Ganzes, als Gruppe von Bergen und Tälern betrachet, vielfach nicht hinwegzulaufen scheint. Man hat diese Erscheinung erklärt durch Annahme zweier gleichgerichter Wellen in Überlagerung, die gleiche konstante Amplitude aber etwas verschiedene Wellenlänge haben, gemäss der Identität

$$u = A\left\{\sin\left(2\pi\frac{V_1 t - x}{\lambda_1}\right) + \sin\left(2\pi\frac{V_2 t - x}{\lambda_2}\right)\right\}$$

$$= 2A\cos\pi\left\{\left(\frac{V_1}{\lambda_1} - \frac{V_2}{\lambda_2}\right)t - \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)x\right\}\sin\left\{2\pi\frac{Vt - x}{\lambda}\right\} \quad (C)$$

$$\text{mit}\quad \frac{V}{\lambda} = \frac{1}{2}\left(\frac{V_1}{\lambda_1} + \frac{V_2}{\lambda_2}\right)\ ;\quad \frac{1}{\lambda} = \frac{1}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right).$$

Dieser Erklärung obiger Erscheinung durch Interferenz und Dispersion stellen wir unsere allgemeine Wellenform (C) gegenüber, die mit ihren zwei wandernden Phasen eine Welle höherer Ordnung darstellt."

This explanation is not a true representa- tion of the mathematics of dispersion. Geometrical dispersion appears as a natural consequence in the solution of many problems involving wave propagation, and the resulting expression usually has a simple physical interpretation. The problem of a source moving along the surface of a liquid has been solved by Lamb[7] and presents the correct picture of this type of water wave propagation.

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6. ibid, 403
7. Lamb, H., Hydrodynamics, 433-434, Dover, (1945)

The quotations that have been given are all from a single article. This was not the first article that Uller wrote on his wave theory, but it is the article in which he formally states and explains his wave theory. Other articles extend his theory to spherical waves and to gravity effect. In other papers he considers surface waves on a free surface and along the interface between two elastic media. Reference to these articles is given in the bibliography.

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"Uber die Front- und Rückengeschwindigkeit von Verzerrungswellen in festen, schweren Körpern", Gerland's Beiträge zur Geophysik, 16, (1926), 219-238

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"Analyse der Planwelle von elementarer Schwankungsform, sowie ihre Verwandung zur augenäherten Wiedergabe einer allegemeineren Wellenform", Gerland. 20, (1928), 123-132

"Wellen-Induktion", Gerland., 19, (1928), 134-146

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Part 2 - On Electromagnetic Surface Waves

ABSTRACT

It is shown by consideration of the outgoing wave condition and the boundary condition at infinity that free electromagnetic surface waves cannot exist along the interface between two semi-infinite media.

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The problem of the propagation of electromagnetic waves from a dipole source situated on the interface between two semi-infinite media, air and earth, was first examined by Sommerfeld[1] and later by Weyl[2]. In the evaluation of the resultant integral,

$$\Pi_1 = 2k_2^2 \int_0^\infty \frac{e^{-\sqrt{\lambda^2-k_1^2}\,z}}{k_1^2\sqrt{\lambda^2-k_2^2}+k_2^2\sqrt{\lambda^2-k_1^2}} J_0(\lambda r)\,\lambda\,d\lambda \qquad (1a)$$

$$\Pi_2 = 2k_1^2 \int_0^\infty \frac{e^{\sqrt{\lambda^2-k_2^2}\,z}}{k_1^2\sqrt{\lambda^2-k_2^2}+k_2^2\sqrt{\lambda^2-k_1^2}} J_0(\lambda r)\,\lambda\,d\lambda \qquad (1b)$$

where Π_1 and Π_2 are the z-components of the Hertzian vector in air and earth respectively and k_1 and k_2 are the wave number in air and earth respectively, Sommerfeld obtained a term corresponding to a free surface wave, i.e., a wave which is propagated along the interface between the two media, which decreases as $r^{-1/2}$ in the direction of propagation, and which decreases exponentially in both directions from the

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1. Sommerfeld, A., Ann. d. Physik 28, 665, (1909)
2. Weyl, H., Ann. d. Physik 60, 481 (1919)

interface. It is of the form

$$\Pi_1 = 2\sqrt{\frac{2\pi i}{r}}\ \frac{k_2^2}{K}\ e^{ipr - \sqrt{p^2-k_1^2}\, z} \tag{2a}$$

$$\Pi_2 = 2\sqrt{\frac{2\pi i}{r}}\ \frac{k_1^2}{K}\ e^{ipr + \sqrt{p^2-k_2^2}\, z} \tag{2b}$$

where $\quad p^2 = \dfrac{k_1^2 k_2^2}{k_1^2 + k_2^2} \quad$ and $\quad K = \dfrac{k_2^2}{\sqrt{p^2-k_1^2}} + \dfrac{k_1^2}{\sqrt{p^2-k_2^2}}$.

Weyl, on the other hand, did not obtain a free surface wave.* Considerable attention has been given to this discrepancy without any clear cut answer having been obtained.

It is the intent of this paper to resolve that difficulty and to show that the surface wave cannot exist. For this purpose an alternative derivation of the fundamental integral is given, following the method of Pekeris[3,4]. Let us take cylindrical coordinates centered at the dipole source as indicated in Figure 1.

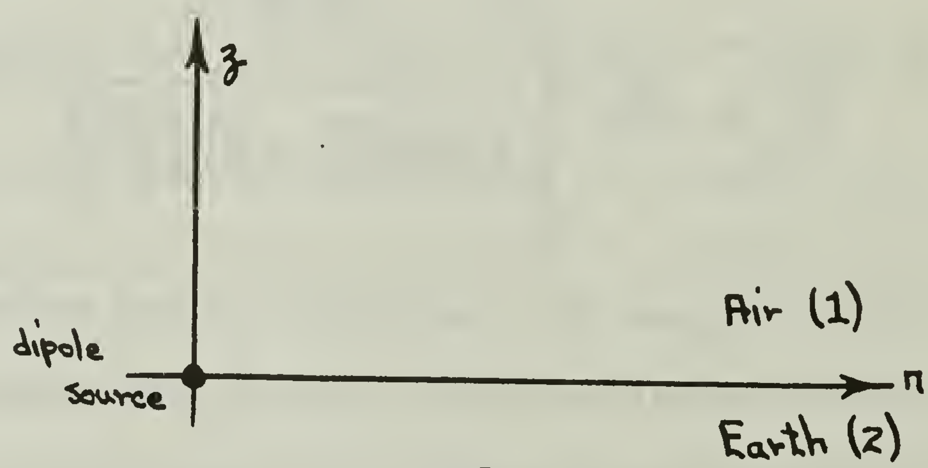

Figure 1

It is known that the radiation from a dipole source can be defined by the z-component of the Hertzian vector Π , where Π must satisfy the

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* A good resume of the two methods of solution is found in Stratton's <u>Electromagnetic Theory</u>, McGraw-Hill, New York, 1941, pp. 573-587.

3. Pekeris, C., G.S.A. Memoir 27, 44, (1948)
4. Pekeris, C., J. Acoust. Soc. <u>18</u>, 296 (1946)

wave equations,

$$\nabla^2 \Pi_1 = \frac{1}{c_1^2}\ddot{\Pi}_1 \tag{3a}$$

and

$$\nabla^2 \Pi_2 = \frac{1}{c_2^2}\ddot{\Pi}_2 \tag{3b}$$

and the boundary conditions at z = 0,

$$k_1^2 \Pi_1 - k_2^2 \Pi_2 = 0 \tag{4a}$$

$$\frac{\partial \Pi_1}{\partial z} - \frac{\partial \Pi_2}{\partial z} = 0 \qquad (r \neq 0) \tag{4b}$$

To satisfy these conditions, solutions of the form

$$\Pi_1 = e^{i\omega t} J_0(\lambda r) F_1(z) G_1(\lambda) \tag{5a}$$

$$\Pi_2 = e^{i\omega t} J_0(\lambda r) F_2(z) G_2(\lambda) \tag{5b}$$

are taken where λ is an arbitrary variable of integration. The condition that there be a dipole source at the origin can now be most easily met by changing the second boundary condition to read

$$\frac{\partial \Pi_1}{\partial z} - \frac{\partial \Pi_2}{\partial z} = Z\lambda \tag{6}$$

for when the expressions (5) are integrated with respect to λ from 0 to ∞, the discontinuity in $\frac{\partial \Pi}{\partial z}$ at $z=0$ becomes proportional to $\int_0^\infty J_0(\lambda r)\,\lambda\, d\lambda$. This function vanishes for every value of r except $r=0$ where it becomes infinite in such a manner that its integral over the plane z = 0 is finite. Substituting (5) into (3) we obtain,

$$\frac{d^2F_1}{dz^2} - \beta_1^2 F_1 = 0 \tag{7a}$$

$$\frac{d^2F_2}{dz^2} - \beta_2^2 F_2 = 0 \tag{7b}$$

where $\beta_1^2 = \sqrt{\lambda^2 - k_1^2}$

and $\beta_2^2 = \sqrt{\lambda^2 - k_2^2}$

The solutions of (7) satisfying the boundary conditions at ∞ will then be

$$F_1 = A_1 e^{-\sqrt{\lambda^2 - k_1^2}\, z} \tag{8a}$$

$$F_2 = A_2 e^{\sqrt{\lambda^2 - k_2^2}\, z}$$

and for the Hertzian vector,

$$\Pi_1 = e^{i\omega t} J_0(\lambda r) e^{-\sqrt{\lambda^2 - k_1^2}\, z} H_1(\lambda)$$

$$\Pi_2 = e^{i\omega t} J_0(\lambda r) e^{\sqrt{\lambda^2 - k_2^2}\, z} H_2(\lambda)$$

It is necessary at this point to make a few comments about the sign convention taken for the radicals β_1 and β_2. β_1 and β_2 will be in general complex. In order to satisfy the boundary conditions at ∞, as given by solutions (9), the sign of the real part of the radicals must always be taken positive. In order to have waves propagating away from the dipole source, the sign of the imaginary part of the radicals must be taken positive. Both of the radicals will then in general be given by a positive real plus a positive imaginary. This condition is of the first importance when the investigation of the resulting integrals is considered.

Substituting equations (9) into the boundary and source conditions at $z = 0$, (4a) and (6), we obtain for H_1 and H_2

$$H_1 = \frac{2\lambda k_2^2}{k_1^2\sqrt{\lambda^2-k_2^2} + k_2^2\sqrt{\lambda^2-k_1^2}} \tag{10a}$$

$$H_2 = \frac{2\lambda k_1^2}{k_1^2\sqrt{\lambda^2-k_2^2} + k_2^2\sqrt{\lambda^2-k_1^2}} \tag{10b}$$

The final expressions for Π_1 and Π_2 will then be

$$\Pi_1 = 2k_2^2 \int_0^\infty \frac{e^{-\sqrt{\lambda^2-k_1^2}\, z}}{k_1^2\sqrt{\lambda^2-k_2^2} + k_2^2\sqrt{\lambda^2-k_1^2}} J_0(\lambda r)\,\lambda\, d\lambda \tag{11a}$$

$$\Pi_2 = 2k_1^2 \int_0^\infty \frac{e^{\sqrt{\lambda^2-k_2^2}\, z}}{k_1^2\sqrt{\lambda^2-k_2^2} + k_2^2\sqrt{\lambda^2-k_1^2}} J_0(\lambda r)\,\lambda\, d\lambda\,. \tag{11b}$$

which are the same as those obtained by Sommerfeld, equations (1).

The evaluation of the integrals (11) is most easily effected by transforming the path of integration to the complex plane and making the substitutions,

$$J_0(\lambda r) = \frac{1}{2}\left[H_0^{(1)}(\lambda r) + H_0^{(2)}(\lambda r)\right]$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are the two Hankel functions and have the properties that $H_0^{(1)}$ vanishes at ∞ in the first quadrant and $H_0^{(2)}$ vanishes at ∞ in the fourth quadrant. The paths of integration for $H_0^{(1)}$ and

$H_o^{(2)}$ are indicated in Figure 2 where k_1 and k_2 are branch points.

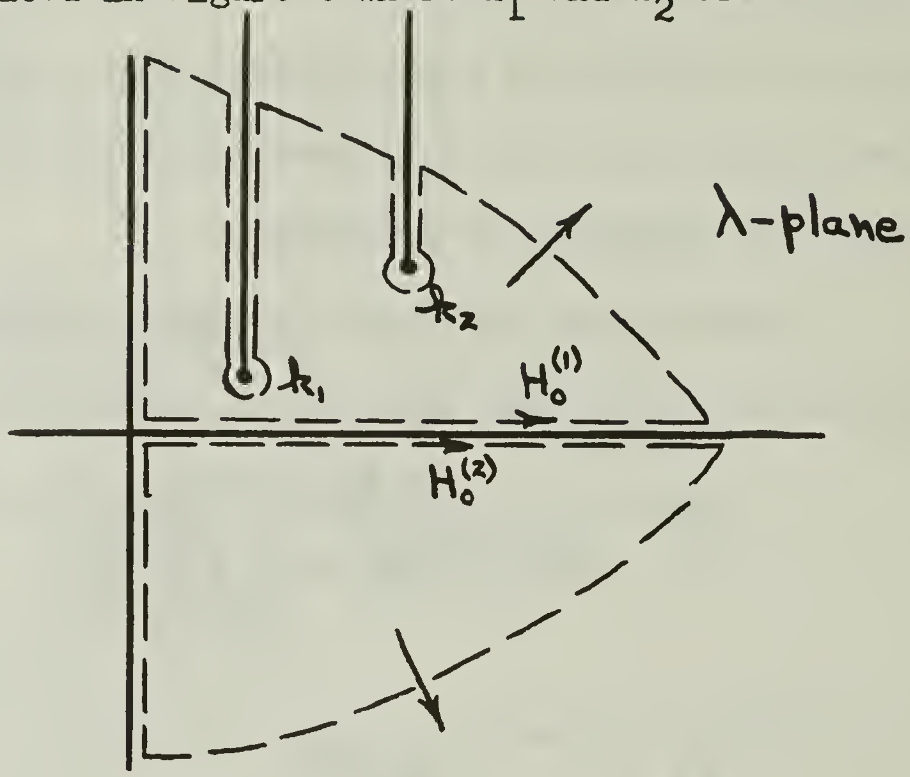

Figure 2

As

$$H_o^{(1)}(iy) = -H_o^{(2)}(-iy)$$

the values of Π_1 and Π_2 will be given by the branch line integrals defined by k_1 and k_2 plus the residues from any poles. The branch line integrals produce the expressions for the body waves in the two media and are identical in Sommerfeld's and Weyl's solution. The interest in this paper is with the existence of any residues which would give rise to additional terms in the final solution and which in Sommerfeld's solution produced the electromagnetic surface wave. Poles in the λ-plane will be located by solutions of the equation

$$k_1^2\sqrt{\lambda^2-k_2^2}+k_2^2\sqrt{\lambda^2-k_1^2} = k_1^2\beta_1 + k_2^2\beta_2 = 0 \tag{12}$$

The values for k_1^2 and k_2^2 are given by

$$k_1^2 = \varepsilon_1\mu_1\omega^2 + i\mu_1\sigma_1\omega$$

$$k_2^2 = \varepsilon_2\mu_2\omega^2 + i\mu_2\sigma_2\omega$$

where ε_1 and ε_2 are the dielectric constants of air and earth respectively, μ_1 and μ_2 the permeabilities, and σ_1 and σ_2 the conductivities. Thus, k_1^2 and k_2^2 will always be given by a positive real part plus a positive imaginary part. As stated previously, β_1 and β_2 will also have positive real and imaginary parts in order to satisfy the outgoing wave condition and the boundary condition at infinity. It is immediately apparent then that no value of λ will satisfy equation (12); and consequently that there are no poles in the complex λ-plane satisfying the physical conditions. There is no contribution corresponding to a surface wave. In the previous solution regard for the sign of the radicals had been lost by squaring equation (12) and solving for λ.

Two papers have been written recently on this same subject, one by Kahan and Eckart[5] and the other by Epstein[6]. It is felt that a few remarks are in order concerning their results. Kahan and Eckart's statement that the surface wave does not satisfy the outgoing wave condition is correct. However, they resolve the difficulty between Sommerfeld's and Weyl's solutions by stating that a saddle point, k_s, has been overlooked in the evaluation of one of the branch line integrals. The inclusion of this saddle point in the integration,

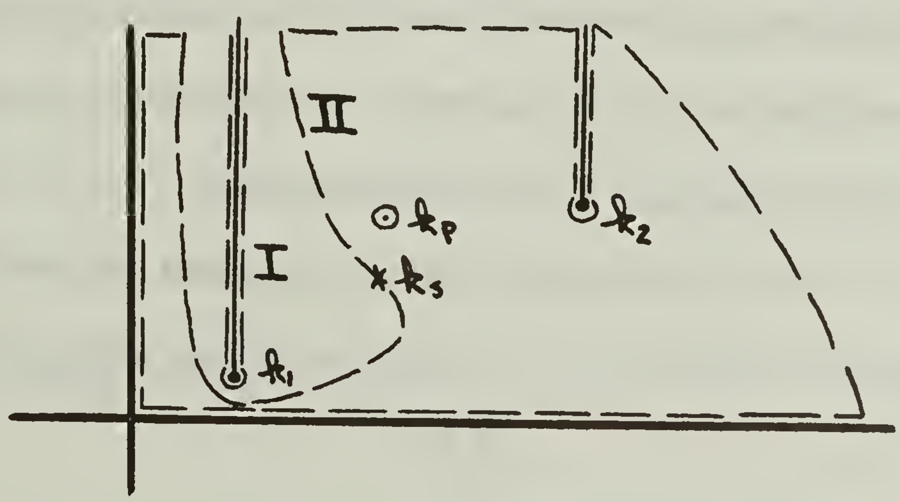

Figure 3

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5. Kahan, T., and Eckart, G., Phys. Rev. 76, 406 (1949)
6. Epstein, P., Proc. Nat. Acad. Sci. 33, 195 (1947)

they say, will produce a term which will balance the surface wave expression obtained from the residue of the pole, k_p. In going from path I (Sommerfeld) to path II (Kahan and Eckart), however, no singularities are passed; and consequently the exact value of the integral along path I must necessarily be the same as that along path II. It is possible that the approximate evaluations will differ by an amount equal to the surface wave expression, but this has not been demonstrated.

Epstein, on the other hand, argues that the path of integration in the λ-plane can be displaced out of the neighborhood of the supposed pole before the boundary conditions are introduced, thus eliminating the surface wave. One questions the validity of this step; for by transforming the path of integration before the introduction of the boundary conditions, the singularities that may occur due to the boundary conditions alone are eliminated. He then examines the conditions necessary to have a free surface wave propagated along the interface but fails to include the restrictions imposed on the signs of the radicals by the boundary condition at infinity and the outgoing wave condition. This problem is reexamined in the next paragraph, where the existence of free electromagnetic surface waves is investigated for the simple case of plane waves. (The solution carries over easily to spherical waves.)

Let us consider the z-component of the Hertzian function propagated along the xy interface in the x-direction (see Figure 4)

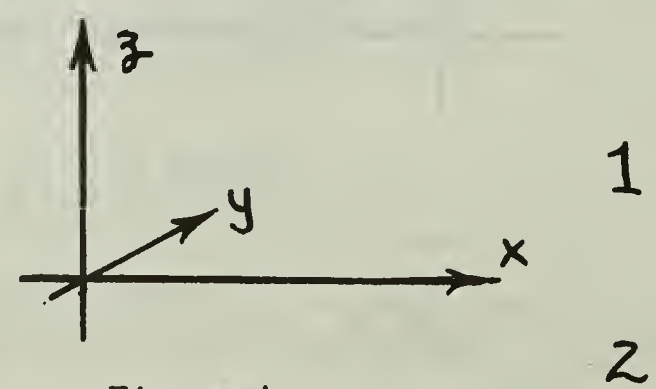

Figure 4

of the form

$$\Pi_1 = f(z)\, e^{ik(x-ct)} \tag{13a}$$

$$\Pi_2 = g(z)\, e^{ik(x-ct)} \tag{13b}$$

satisfying the wave equations

$$\ddot{\Pi}_1 = c_1^2 \nabla^2 \Pi_1 \tag{14a}$$

$$\ddot{\Pi}_2 = c_2^2 \nabla^2 \Pi_2 \tag{14b}$$

Substituting equations (13) into equations (14), a second order differential equation in z is obtained, the solution of which is

$$f = A e^{-ik\left(\frac{c^2}{c_1^2}-1\right)^{1/2} z} + B e^{ik\left(\frac{c^2}{c_1^2}-1\right) z} \tag{15a}$$

$$g = C e^{-ik\left(\frac{c^2}{c_2^2}-1\right)^{1/2} z} + D e^{ik\left(\frac{c^2}{c_2^2}-1\right)^{1/2} z} \tag{15b}$$

where A = D = 0 in order to satisfy the boundary conditions at infinity. The boundary conditions at z = 0 are given by equations (3). Substituting equations (15) into equations (13) and into the boundary conditions, the following equations for the amplitudes are obtained:

$$k_1^2 B - k_2^2 C = 0 \tag{16a}$$

$$\left(\frac{c^2}{c_1^2}-1\right)^{1/2} B + \left(\frac{c^2}{c_2^2}-1\right)^{1/2} C = 0 . \tag{16b}$$

In order for the above equations to have a solution, the determinant of their coefficients must vanish.

$$k_1^2\left(\frac{c^2}{c_2^2}-1\right)^{1/2}+k_2^2\left(\frac{c^2}{c_1^2}-1\right)^{1/2}=0 \tag{17}$$

Substituting

$$c_1=\frac{\omega}{k_1}, \quad c_2=\frac{\omega}{k_2}, \text{ and } \quad c=\frac{\omega}{k}$$

gives

$$k_1^2\sqrt{\lambda^2-k_2^2}+k_2^2\sqrt{\lambda^2-k_1^2}=0 \tag{18}$$

which is identical with equation (12) and as before will not have a solution satisfying the physical conditions of the problem. It is concluded that free electromagnetic surface waves cannot exist along the interface between two semi-infinite media.

Part 3 - Normal Mode Propagation in Three Layered Liquid Half-Space by Ray Theory

ABSTRACT

The fundamental integral for narmal mode propagation in a three liquid layered half-space is derived by multiple reflections and the physical significance of the characteristic equation is discussed.

* * * * * * * *

Pekeris[1] has shown how the fundamental integral for two liquid layer normal mode propagation can be derived by ray theory. Press and Ewing[2] have discussed the physical significance of normal mode propagation in the two layered case. It is the purpose of this paper to derive the fundamental integral and discuss the physical significance of normal mode propagation in the three layered case.

A spherical wave may be represented by the following integral:

$$e^{-\frac{\omega}{c}R} = \int_0^\infty J_0(kr)\,\frac{k\,dk}{i\beta}\,e^{-i\beta(d-z)} \tag{1}$$

where $R = \sqrt{r^2+(d-z)^2}$

$\beta = \sqrt{\frac{\omega^2}{c^2}-k^2}\,, \quad k < \frac{\omega}{c}$

$\quad = -i\sqrt{\frac{\omega^2}{c^2}-k^2}\,, \quad k > \frac{\omega}{c}$

$\omega =$ circular frequency of the source

$c =$ veoloity of sound propagation

$r =$ horizontal range

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1. Pekeris, C., Geol. Soc. Amer., Mem. 27 (1948)
2. Press, F. and Ewing, M., A. G. U. Trans., <u>29</u>, 163-174 (1948)

$d =$ depth of the source

and $z =$ depth of the receiver.

The integral merely represents the manner in which spherical source is built up from the summation of plane wave about a point. k represents the wave number as measured along the horizontal surface, and β_1 represents the wave number as measured along the vertical. The summation is taken over real and imaginary angles of incidence. In a reflected wave, see Figure 1, the integral becomes

$$\int_0^\infty J_0(kr)\,\frac{k\,dk}{i\beta}\,K e^{-i\beta(d+z)} \qquad (2)$$

where K is the plane wave reflection coefficient. The discussion of integrals of this type generalized to an impulsive point source has been carried out by Pekeris[3] and by Arons and Yennie[4].

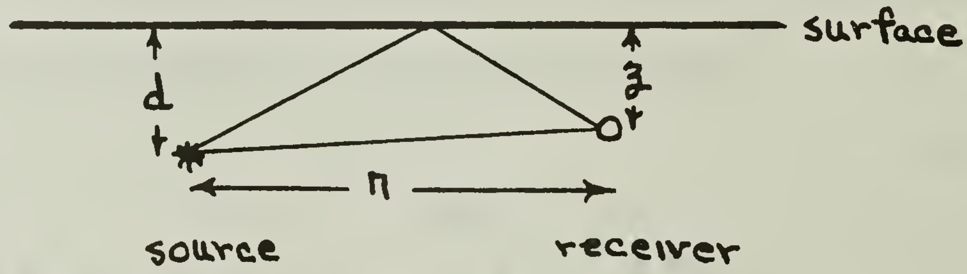

Figure 1

The fundamental integral for propagation in a layered medium can then be obtained by the summation of multiple reflections from a plane wave source and the generalization of this summation to a point source. A plane wave source is considered to consist of two sets of outgoing waves - one upgoing and the other downgoing and both at the

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3. Pekeris, C., loc. cit.
4. Arons, A., and Yennie, D., Jour. Acoust. Soc. 22, 231-237 (1950).

same angle of incidence with the horizontal. The sum of the z-component of the multiple reflections is given by equation (3), see Figure 2, where

$K =$ reflection coefficient from medium 1 to medium 2

$K' =$ reflection coefficient from medium 2 to medium 1

$L =$ reflection coefficient from medium 2 to medium 3

and $TT =$ product of the refraction coefficient from medium 1 to medium 2 and the refraction coefficient from medium 2 to medium 1

The expression in the first brackett is the sum of the direct ray and the first reflected ray. The first term in the expression

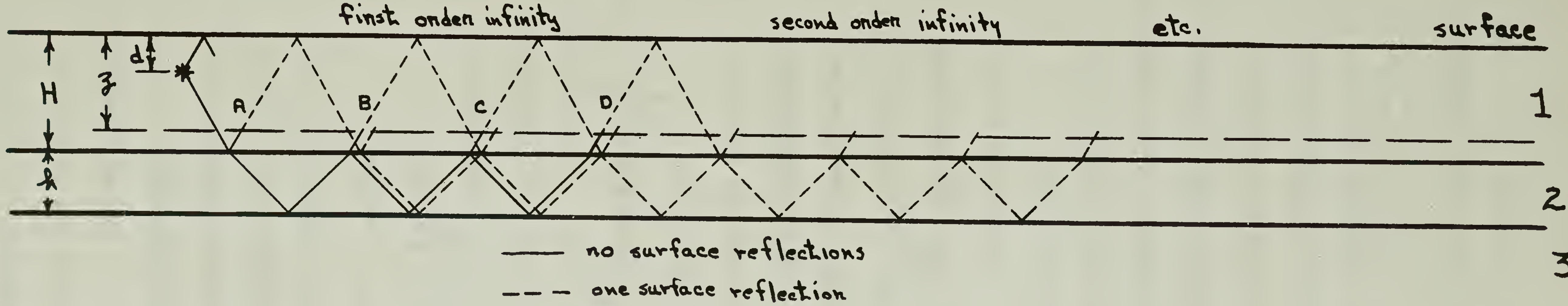
$$\left\{ K + LTTe^{-i2\beta_2 h} + K'L^2TTe^{-i4\beta_2 h} + K'^2L^2TTe^{-i6\beta_2 h} + \cdots \right\} e^{-i2\beta_1 (H-z)} \tag{4}$$

is the ray reflected from the 1-2 interface. The second term is the ray refracted from 1 to 2, reflected at the 2-3 interface, and re-fracted back into medium 1. The third term represents the ray which has undergone one more reflection in the second layer and then been refracted back into medium 1. This series of rays is labelled A, B, C, D in Figure 2.

The second term in the third brackett multiplied by the first two bracketts represents the effects of the set of rays discussed above after they have gone through one reflection at the surface. Some of these rays are indicated in Figure 2. The next term gives the effects of this set of rays after they have gone through another surface reflection and so on for the succeeding terms.

In normal mode theory the approximation is made that each finite sum of multiple reflections can be represented by its appropriate infinite sum. This approximation is now applied to expression (3).

The reflection and refraction coefficients are given by.



down going ray　up going ray

$$\left[e^{-i/\beta_1(z-d)} - e^{-i/\beta_1(z+d)}\right] \times \left[1 + e^{-i2/\beta_1(H-z)}\left\{K + LTTe^{-i2/\beta_2 h} + K'L^2TTe^{-i4/\beta_2 h} + K'^2L^3TTe^{-i6/\beta_2 h} + \cdots\right\}\right]$$

$$\times\left[1 - e^{-i2/\beta_1 H}\left\{K + LTTe^{-i2/\beta_2 h} + K'L^2TTe^{-i4/\beta_2 h} + K'^2L^3TTe^{-i6/\beta_2 h} + \cdots\right\}\right. \qquad (3)$$

$$+ e^{-i4/\beta_1 H}\left\{K + LTTe^{-i2/\beta_2 h} + K'L^2TTe^{-i4/\beta_2 h} + \cdots\right\}\left\{K + LTTe^{-i2/\beta_2 h} + K'L^2TTe^{-i4/\beta_2 h} + \cdots\right\}$$

$$\left.\bullet\ \cdots\right]$$

Figure 2

$$K = \frac{\beta_1 - b\beta_2}{\beta_1 + b\beta_2} \tag{5}$$

$$K' = \frac{b\beta_2 - \beta_1}{b\beta_2 + \beta_1} \tag{6}$$

$$L = \frac{\beta_2 - g\beta_3}{\beta_2 + g\beta_3} \tag{7}$$

$$TT = \frac{4b\beta_1\beta_2}{(\beta_1 + b\beta_2)^2} = 1 - K^2 = 1 - K'^2 \quad . \tag{8}$$

where $b = \rho_1/\rho_2$

$g = \rho_2/\rho_3$

$\rho_1 =$ density of medium 1

$\rho_2 =$ density of medium 2

$\rho_3 =$ density of medium 3.

Then, the series in the parentheses of equation 3 become

$$K + LTTe^{-i2\beta_2 h} + K'L^2TTe^{-i4\beta_2 h} + K'^2L^3TTe^{-i6\beta_2 h} + \cdots$$

$$= -K' + (L - K'^2L)e^{-i2\beta_2 h} + (K'L^2 - K'^3L^2)e^{-i4\beta_2 h} + (K'^2L^3 - K'^4L^3)e^{-i6\beta_2 h} + \cdots \tag{9}$$

$$= \left(-K' + Le^{-i2\beta_2 h}\right)\left(1 + K'Le^{-i2\beta_2 h} + K'^2L^2e^{-i4\beta_2 h} + K'^3L^3e^{-i6\beta_2 h} + \cdots\right)$$

$$= \frac{-K' + Le^{-i2\beta_2 h}}{1 - K'Le^{-i2\beta_2 h}} = \frac{K + Le^{-i2\beta_2 h}}{1 + KLe^{-i2\beta_2 h}}$$

And substituting (5), (6), (7), and (8) into (9), one obtains

$$\frac{K + Le^{-i2\beta_2 h}}{1 + KLe^{-i2\beta_2 h}} = \frac{R\beta_1 - b\beta_2}{R\beta_1 + b\beta_2}$$

where $\mathcal{R} = \dfrac{1 + L e^{-i2\beta_2 h}}{1 - L e^{-i2\beta_2 h}}$. The third brackett is a geometric series in powers of $\left(- \dfrac{\mathcal{R}\beta_1 - b\beta_2}{\mathcal{R}\beta_1 + b\beta_2} e^{-i2\beta_1 H} \right)$ so that expression (3) will be given by

$$\left[e^{-i\beta_1 (z-d)} - e^{-i\beta_1 (z+d)} \right] \frac{1 + \dfrac{\mathcal{R}\beta_1 - b\beta_2}{\mathcal{R}\beta_1 + b\beta_2} e^{-i2\beta_1 (H-z)}}{1 + \dfrac{\mathcal{R}\beta_1 - b\beta_2}{\mathcal{R}\beta_1 + b\beta_2} e^{-i2\beta_1 H}}$$

$$= 2i \sin\beta_1 d \, e^{-i\beta_1 z} \frac{(\mathcal{R}\beta_1 + b\beta_2) + (\mathcal{R}\beta_1 - b\beta_2) e^{-i2\beta_1 (H-z)}}{(\mathcal{R}\beta_1 + b\beta_2) + (\mathcal{R}\beta_1 - b\beta_2) e^{-i2\beta_1 H}} \tag{10}$$

$$= 2i \sin\beta_1 d \, \frac{\mathcal{R}\beta_1 \cos\beta_1 (H-z) + i b\beta_2 \sin\beta_1 (H-z)}{\mathcal{R}\beta_1 \cos\beta_1 H + i b\beta_2 \sin\beta_1 H}$$

The integral representation for a spherical point source will then be

$$\phi = 2 \int_0^\infty J_0(kr) k \, dk \, \frac{\mathcal{R}\beta_1 \cos\beta_1 (H-z) + i b\beta_2 \sin\beta_1 (H-z)}{\mathcal{R}\beta_1 \cos\beta_1 H + i b\beta_2 \sin\beta_1 H} \tag{11}$$

which agrees with the expression given by Pekeris[5] if one realizes that $\mathcal{R}$ can also be expressed as

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5. Pekeris, C., loc. cit.

$$R = i\,\frac{g\beta_3 \tan\beta_2 h - i\beta_2}{g\beta_3 + i\beta_2 \tan\beta_2 h}$$

In the formal solution of the integral (11) the normal modes occur at the poles of the integrand. The equation determining these poles is the dispersion equation for the phase velocity and is called the characteristic equation of the normal mode system. The physical significance of the characteristic equation for the two layered system has been discussed by Press and Ewing[6]. The significance of the three layered equation is discussed here.

The poles of the integrand occur at solutions of the equation,

$$R\beta_1 \cos\beta_1 H + i b\beta_2 \sin\beta_1 H = 1 + \frac{R\beta_1 - b\beta_2}{R\beta_1 + b\beta_2}\, e^{-i2\beta_1 H}$$

$$= 1 + \left(K + LTTe^{-i2\beta_2 h} + K'L^2TTe^{-i4\beta_2 h} + K'^2L^3TTe^{-i6\beta_2 h} + \cdots\right) e^{-i2\beta_1 H} \quad (12)$$

or

$$1 = -\left(K + LTTe^{-i2\beta_2 h} + K'L^2TTe^{-i4\beta_2 h} + K'^2L^3TTe^{-i6\beta_2 l} + \cdots\right) e^{-i2\beta_1 H} \quad (13)$$

The equation says that the normal modes occur at those places where the primary ray is in phase with the sum of the secondary rays which have not undergone another reflection from the free surface and where the amplitude of the sum of the secondary rays is equal to the primary.

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6. Press, F., and Ewing, M., loc. cit.

The ray pattern for the first mode as described by equation (13) is indicated in Figure 3.

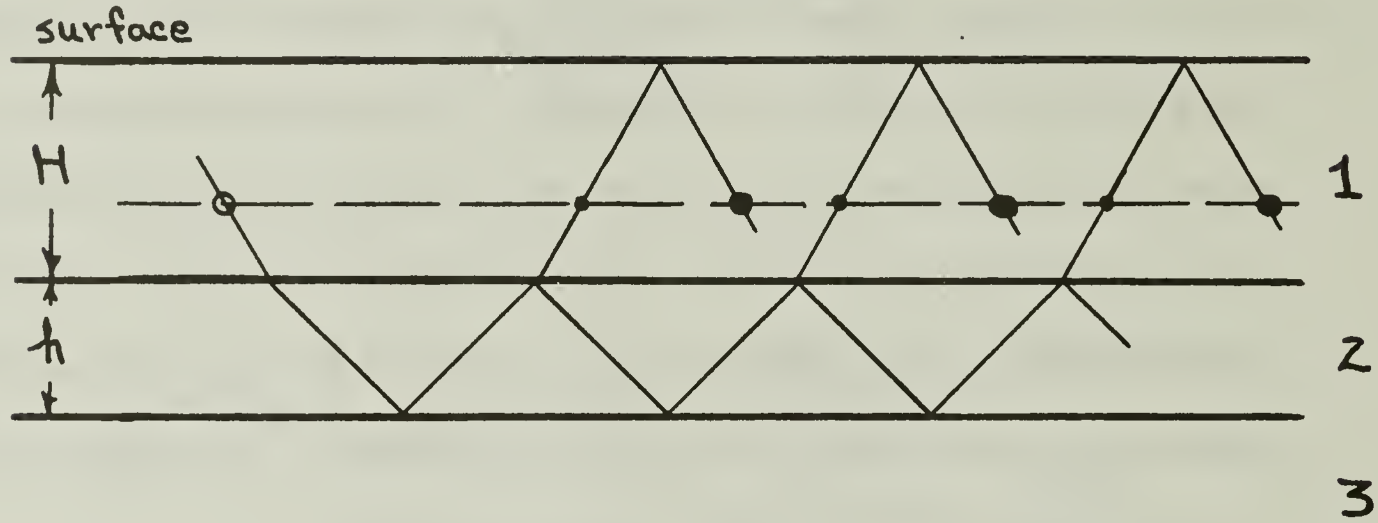

Figure 3

In the two layered case the condition was that the reflected ray be in phase with the primary ray, see Figure 4.

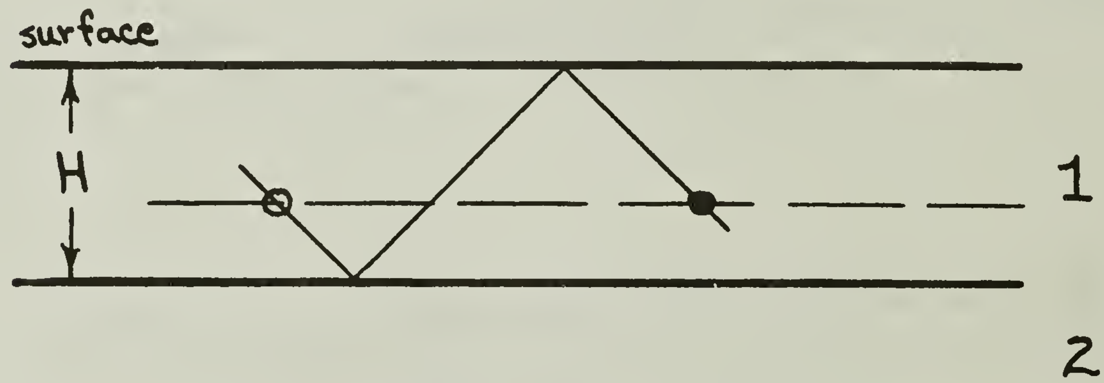

Figure 4

In the three layered case the effect of the partial refraction at the first interface must be included. (It is to be remembered that the expressions obtained from the integral and discussed here apply to the z -component of the wave. A similar discussion holds for the normal to the wave.)

The higher modes occur at increasing values of k corresponding to multiples of 2π in the phase. The ray patterns for these modes will be similar to Figure 3. The cut off frequency occurs when the sum of the amplitudes of the secondary rays does not equal unity, i.e., when there is partial refraction into medium 3. At the upper end of the spectrum a frequency is passed beyond which the propagation takes place solely in the first layer. This corresponds to critical reflection at the 1-2 interface.

It is hoped that the method of multiple reflections used here for the interpretation of normal modes in the three liquid layer case can be utilized to obtain the characteristic equation for more complicated problems which, as yet, have not been solved by formal methods.